CORRESPONDENCE

Comments on "The Over-Relaxation Factor in the Numerical Solution of the Omega Equation"

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Stuart and O'Neill [1] showed results of tests for the optimum over-relaxation coefficients in numerical solutions of the omega equation using one-, two-, and three-dimensional forms of the equation. They compared their observed results with the theoretical estimate obtained from an equation given by Miyakoda [2]. One of their conclusions is as follows: for the one- and two-dimensional cases, the optimum over-relaxation coefficients they obtained from tests agree very well with the theoretical estimate. In the three-dimensional cases, however, the observed over-relaxation coefficients are found to be less than the value given by the theory. In other words, cut-off in the empirically determined over-relaxation factor, α, is less than α_{opt} estimated from the analysis. They state that the effect of vertically variable σ, the static stability, resulted in the observed value of α_{opt} being much below the theoretical value which was obtained assuming constant σ.

I was very interested in this conclusion and recalculated the theoretical α_{opt}. Because I obtained some different results, I am making the following comments. According to my calculation, not only in the one- and two-dimensional case, but also in the three-dimensional case, the theoretical values agree very well with Stuart and O'Neill's observed values. Adopting their notation, we may write Miyakoda's equations in the following way:

$$\alpha=\frac{1+V}{K'_{ijp}}=\frac{1+V}{4+2K_{ijp}}$$

$$K'_{ijp}=4+2K_{ijp}$$

$$K_{ijp}=\frac{a^2f_0^2}{m^2\sigma(\Delta p)^2}$$

$$V=C-\sqrt{C^2-1}$$

$$C=\left[\left(\frac{4+2K_{ijp}}{(2+K_{ijp})\tau}\right)^2\Big/2\right]-1=\frac{2}{\tau^2}-1$$

$$\tau=\frac{1}{2+K_{ijp}}\left(\cos\frac{\pi}{N_{x-1}}+\cos\frac{\pi}{N_{y-1}}+K_{ijp}\cos\frac{\pi}{N_{p-1}}\right)$$

where V is the optimum over-relaxation coefficient. Approximate calculation yielded:

$\alpha=0.420$ in the two-dimensional case

$\alpha=0.308$ in the three-dimensional case with $a=2°$ lat. grid interval

$\alpha=0.227$ in the three-dimensional case with $a=3°$ lat. grid interval.

The other parameters are the same with those employed by Stuart and O'Neill. Comparing my results with the values given in their tables 2 and 3, I found that my results agree quite well with their observed values but differ greatly from their theoretical values in the three-dimensional cases.

Mr. H. Kato, Japan Meteorological Agency, also examined Stuart and O'Neill's results independently using an electronic computer HITAC 5020 E/F. Table 1 summarizes the comparison of Kato's results with Stuart and O'Neill's values. The one- and two-dimensional cases are omitted. In his calculation he used the same parameters that Stuart and O'Neill employed. From these results, I conclude that also in the three-dimensional case, Miyakoda's equation gives a quite reasonable estimate of the optimum over-relaxation coefficient. The effect of variable σ has less significant influence, at least in the case Stuart and O'Neill treated. Indeed σ does not vary greatly in the troposphere and changes abruptly to a value 10 times larger in the stratosphere. But a large value of σ makes K_{ijp} small which makes α large. Therefore if the effect of variable σ has considerable influence on the determination of α, I speculate the effect should be to make the observed value much larger than the theoretical estimate. Indeed $\sigma=44$ in MTS units gives $\alpha=0.414$, with $a=2°$ lat. grid interval, $N_x=N_y=18$, $N_p=6$, and the other parameters being identical with Stuart and O'Neill's.

TABLE 1.—*Comparison of the optimum over-relaxation factors*

Grid size	Stuart and O'Neill [1]				Kato		
	α_{opt}		Grid point		α_{opt}	Grid point	
	Theory	Observed	$N_x=N_y$	N_p	Theory	$N_x=N_y$	N_p
1°	0.45	0.400	35	6	0.4019	35	6
1.5°	.44	.250	23	6	.3515	23	6
2°	.42	.300	17	6	.3043	17	6
2°	.42	.320	18	6	.3055	18	6
3°	.33	.225	11	6	.2250	11	6

REFERENCES

1. D. W. Stuart and T. H. R. O'Neill, "The Over-Relaxation Factor in the Numerical Solution of the Omega Equation," *Monthly Weather Review*, vol. 95, No. 5, May 1967, pp. 303–307.
2. K. Miyakoda, "Test of Convergence Speed of Iterative Methods for Solving 2 and 3 Dimensional Elliptic-Type Differential Equations," *Journal of the Meteorological Society of Japan*, Ser. II, vol. 38, No. 2, Apr. 1960, pp. 107–124.

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Reply

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Capt. O'Neill and I are indebted to Mr. Yamagishi for his above comments pointing out the error in our paper [3]. Indeed the α_{opt} values from the theory as given in table 1 of his note should replace those given in table 3 of our paper. I must accept the responsibility for this error. After receiving Mr. Yamagishi's correspondence I traced this error to the use of an incorrect value of f_0^2 in the expression for K_{ijp}.

As pointed out by Yamagishi and as seen in his table 1, it appears that our observed α_{opt} values agree well with the theory as developed by Miyakoda [1] even for the three-dimensional (3-D) case. However, some caution must be exerted here. Miyakoda's analysis is developed for K_{ijp} constant and in the cases considered by Stuart and O'Neill [3] (i.e., the quasi-geostrophic omega equation) K_{ijp} varies with pressure due to the variation of static stability ($\sigma=-(\alpha/\theta)\partial\theta/\partial p$). Focusing only on the 2° grid case with $N_x=N_y=18$, $N_p=6$ and using σ for the standard atmosphere, the correct α_{opt} via the theory is: $\alpha_{opt}=0.237$ for $\sigma=1.178$ MTS units (at 800 mb.), $\alpha_{opt}=0.306$ for $\sigma=2.0$ MTS units (at 600 mb.), $\alpha_{opt}=0.352$ for $\sigma=4.0$ MTS units (at 440 mb.) and $\alpha_{opt}=0.414$ for $\sigma=44$ MTS units (at 200 mb.). Table 3 of Stuart and O'Neill [3] shows this case to have an observed $\alpha_{opt}=0.320$ with a sharp cutoff near $\alpha=0.350$. Hence, for the relaxation scheme used by Stuart and O'Neill for the solution of the 3-D omega equation a choice of $\sigma>4$ MTS units would have yielded an α_{opt} that led to nonconvergence. In this case the choice of the appropriate σ probably is not too difficult since only one level—200 mb.—had a very high σ and the tropospheric σ could be easily argued to be the most appropriate to yield a theoretical α_{opt} quite close to the observed α_{opt}. Actually the observed α_{opt} falls in the range of Miyakoda's theoretical α_{opt} corresponding to the range of σ but with a weighting toward the lower α's since more levels have lower σ's.

In an earlier correspondence (Stuart [2]), I reported on the extension of the model for the 2° grid case to $N_p=11$ (i.e. $\Delta p=10$ cb.) yet with $N_x=N_y=18$ as before. Miyakoda's theory gives the following values for the optimum over-relaxation factor: $\alpha_{opt}=0.128$ for $\sigma=0.944$ MTS units, $\alpha_{opt}=0.202$ for $\sigma=2.0$ MTS units, $\alpha_{opt}=0.272$ for $\sigma=4.0$ MTS units, and $\alpha_{opt}=0.418$ for $\sigma=206$ MTS units. (The first and last values correspond to σ at 900 and 100 mb. in the standard atmosphere.) In the actual solution of the omega equation, σ for the standard atmosphere was employed at all levels yielding an observed $\alpha_{opt}=0.15$ with a sharp cutoff near $\sigma=0.20$. Again we see that α_{opt} calculated via Miyakoda's theory has a wide variation depending on the σ value but our relaxation scheme yields an observed α_{opt} well within this variation and heavily weighted toward the lower (tropospheric) σ values. Note that nonconvergence would have occurred if we used an α_{opt} based on $\sigma>2$ MTS units.

The above comments and the observed results presented by Stuart and O'Neill [3] suggest some changes in our earlier conclusions concerning the α_{opt} value of the 3-D omega equation. Our observed optimum over-relaxation factors agree better with the limited theory than first thought and we now must definitely conclude that Miyakoda's [1] limited analysis is quite useful for selecting the range of α_{opt} for the 3-D omega equation. Since Miyakoda's theory shows α_{opt} to be quite sensitive to the stability factor, σ, in the 3-D omega equation, it is suggested to choose α_{opt} on the low side of the range of Miyakoda's theoretical α_{opt} values as determined using the range of σ appropriate to the problem. For some σ values the α_{opt} as determined by Miyakoda's analysis may lead to nonconvergence when employed in the quasi-geostrophic omega equation. Finally, the observed sharp cutoff for α just larger than α_{opt} is still an important feature of our observed α curves.

REFERENCES

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2. D. W. Stuart, "Reply" [to Correspondence by James J. O'Brien], *Monthly Weather Review*, vol. 96, No. 2, Feb. 1968, p. 104.
3. D. W. Stuart and T. H. R. O'Neill, "The Over-Relaxation Factor in the Numerical Solution of the Omega Equation," *Monthly Weather Review*, vol. 95, No. 5, May 1967, pp. 303–307.

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